

A note on the minimum skew rank of powers of paths

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Abstract

The minimum skew rank of a simple graph G is the smallest possible rank among all real skew symmetric matrices, whose (i, j) -entry (for $i \neq j$) is nonzero whenever ij is an edge in G and is zero otherwise. In this paper we study the problem of minimum skew rank of powers and strict powers of paths.

Keywords. minimum skew rank, path, (strict) power of a graph.

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1 Introduction

The minimum rank problem of a graph G calls for the computation of the smallest possible rank among the matrices with a specific property (symmetric, skew-symmetric, Hermitian, positive definite), described by the graph G and having entries in a given field \mathbb{F} . This problem has received considerable attention in the last few years (see for example [3, 4, 10, 13, 15]). Observe that many of these articles contain solutions to the minimum rank problem for special classes of graphs, matrices, or fields (see [2, 5, 6, 7, 8, 9, 11]). We have selected to work with skew-symmetric matrices over \mathbb{R} , and the graphs under consideration are two different kinds of powers of paths.

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The minimum rank of powers and strict powers of paths and trees was researched by the Minimum Rank Group at the AIM Workshop [1], the results of this study appearing in [9]. The minimum skew rank problem was researched by the Minimum Skew Rank Group at the IMA Workshop [16], with the first results appearing in [2], and further results in [11]. Throughout this paper, we adopt the notation and terminology from [2], [9], [17], and [14].

A *graph* is a pair $G = (V_G, E_G)$, where V_G is the (finite, nonempty) set of vertices of G and E_G is the set of edges, where an edge is an unordered pair of vertices. All the graphs in this paper are *simple graphs* that is, all graphs are loop-free and undirected. The *order of a graph* G , denoted $|G|$, is the number of vertices of G . If $e = uv \in E_G$, we say that u and v are *endpoints* of e ; we also say that u and v are *adjacent*. Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic*, and we write $G \cong G'$, whenever there exist bijections $\phi : V \rightarrow V'$ and $\psi : E \rightarrow E'$, such that $v \in V$ is an endpoint of $e \in E$ if and only if $\phi(v)$ is an endpoint of $\psi(e)$. A *subgraph* of a graph G is a graph H such that $V_H \subseteq V_G$ and $E_H \subseteq E_G$; the graph $G - e$ denotes the subgraph $(V_G, E_G \setminus \{e\})$ of G . If $W \subseteq V_G$ and $E' = \{uv : u, v \in W, uv \in E_G\}$, the graph (W, E') is referred to as the *subgraph of G induced by W* . The subgraph of G induced by $V_G \setminus \{v\}$ is denoted by $G - v$. A *path* on n vertices is the graph $P_n = (\{v_1, v_2, \dots, v_n\}, \{e_i : e_i = v_i v_{i+1}, 1 \leq i \leq n-1\})$. A graph G , is *connected* if for every pair $u, v \in V_G$, there is a path joining u with v . A *walk of length r* in a graph (V, E) is an alternating sequence: $v_{i_0}, e_{i_1}, v_{i_1}, e_{i_2}, \dots, v_{i_r}, e_{i_r}$, of vertices, $v_{i_j} \in V$, and edges $e_{i_j} \in E$, (not necessarily distinct), such that $v_{i_{j-1}}$ and v_{i_j} are the endpoints of e_{i_j} , for $j = 1, 2, \dots, r$. A *complete graph* is a graph whose vertices are pairwise adjacent, a complete graph on n vertices is denoted by K_n . A graph G is *bipartite* if $V_G = X \cup Y$, with $X \cap Y = \emptyset$, and such that each edge of G has one endpoint in X and the other in Y . A *complete bipartite graph* is a bipartite graph in which each vertex in X is adjacent to all the vertices in Y ; a complete bipartite graph is denoted by K_{n_1, n_2} , where $|X| = n_1$ and $|Y| = n_2$. The *union* of graphs G_1, G_2, \dots, G_k , denoted $\bigcup_{i=1}^k G_i$, is the graph $(\bigcup_{i=1}^k V_i, \bigcup_{i=1}^k E_i)$.

A matrix $A \in \mathbb{R}^{n \times n}$ is *skew-symmetric* if $A^T = -A$; note that the diagonal elements of a skew-symmetric matrix are zero. An $n \times n$ skew-symmetric matrix, $A = [a_{ij}]$ is a *band matrix of bandwidth p* if $a_{i, p+i} \neq 0, 1 \leq i \leq n-p$. The *graph of A* , denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \dots, n\}$ and edges $\{ij : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Let $\mathcal{S}^-(G) = \{A \in \mathbb{R}^{n \times n} : A^T = -A, \mathcal{G}(A) = G\}$ be the set of skew-symmetric matrices described by a graph G . The *minimum skew rank of a graph G* is defined as $\text{mr}^-(G) = \min\{\text{rank}(A) : A \in \mathcal{S}^-(G)\}$.

Definition 1.1. [12, p.281] Let r be a positive integer and $G = (V_G, E_G)$ a graph. The graph G to the power r is the graph $G^r = (V_G, E_{G^r})$, where $ij \in E_{G^r}$ if and only if there is a walk in G from vertex i to vertex j of length at most r .

Definition 1.2. Let r be a positive integer and $G = (V_G, E_G)$ a graph. The graph G to the strict power r is the graph $G^{(r)} = (V_G, E_{G^{(r)}})$, where $ij \in E_{G^{(r)}}$ if and only if there is a walk in G from vertex i to vertex j of length exactly r .

The following results are can be found in [2], where it was established that, in general minimum (symmetric) rank and minimum skew rank cannot be compared, in this paper we establish that if G is a power of a path or a strict power of a path, then $\text{mr}(G) \leq \text{mr}^-(G)$.

Observation 1.3. Let G be a graph.

1. If H is an induced subgraph of G , then $\text{mr}^-(H) \leq \text{mr}^-(G)$.
2. If G has connected components G_1, \dots, G_k , then $\text{mr}^-(G) = \sum_{i=1}^k \text{mr}^-(G_i)$.
3. If $G = \bigcup_{i=1}^k G_i$, then $\text{mr}^-(G) \leq \sum_{i=1}^k \text{mr}^-(G_i)$.

Proposition 1.4. [2, Proposition 4.1] For a path P_n on n vertices,

$$\text{mr}^-(P_n) = \begin{cases} n & \text{if } n \text{ is even;} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 1.5. [2, Theorem 2.1] Let G be a connected graph with $|G| \geq 2$ and let F be an infinite field. Then the following are equivalent:

1. $\text{mr}^-(F, G) = 2$,
2. $G = K_{n_1, n_2, \dots, n_t}$ for some $t \geq 2$, $n_i \geq 1$, $i = 1, \dots, t$,
3. G does not contain P_4 nor the paw as an induced subgraph.

2 Minimum Skew Rank of Powers of Paths

The connection between $\text{mr}^-(P_n^r)$ and $\text{mr}^-(P_n^{(r)})$ is not made clear in [9]. However, we have found that when r is even (see Theorem 2.7), these two quantities are closely related. For the remainder of our discussion we denote the set of vertices of P_n by $\{1, 2, \dots, n-1, n\}$, where 1 and n are the pendant vertices, and $ij \in E_{P_n}$ if and only if $|i - j| = 1$. The following lemma contains some needed results, we omit the proof.

Lemma 2.1. For a positive integer m , with $1 \leq m \leq n$, and $i \in \{1, 2, \dots, n-m+1\}$, the induced subgraph of P_n^r $\left(P_n^{(r)}, \text{ respectively}\right)$ on the set of vertices $\{i, i+1, \dots, i+m-1\}$ is isomorphic to P_m^r $\left(P_m^{(r)}, \text{ respectively}\right)$.

2.1 Minimum Skew Rank of Usual Powers of Paths

We know that $\text{mr}^-(P_3) = \text{mr}^-(P_2) = 2$, and that $P_2^r \cong K_2$, $P_3^r \cong K_3$, for $r \geq 2$, thus $\text{mr}^-(P_3^r) = \text{mr}^-(P_2^r) = 2$.

Theorem 2.2. *If n and r are positive integers, with $n \geq 4$, then*

$$\text{mr}^-(P_n^r) = \begin{cases} n-r & \text{if } 1 \leq r \leq n-3 \text{ and } n-r \text{ is even,} \\ n-r+1 & \text{if } 1 \leq r \leq n-3 \text{ and } n-r \text{ is odd,} \\ 2 & \text{if } r \geq n-2. \end{cases}$$

Proof. Recall ([9]) that the graph P_n^{n-2} is a complete multipartite graph isomorphic to $K_n - e$, where e is the edge $1n$. When $r \geq n-1$, the graph P_n^r is a complete graph isomorphic to K_n . Thus if $r \geq n-2$, it follows from Theorem 1.5, that $\text{mr}^-(P_n^r) = 2$.

Let $1 \leq r \leq n-3$, and observe that each matrix in $\mathcal{S}^-(P_n^r)$ is the sum of $r+1$ band matrices of bandwidths $1, 2, \dots, r+1$ and zero diagonal, thus the upper left $(n-r) \times (n-r)$ submatrix is lower-triangular with nonzero diagonal ([14]), and we have $\text{mr}^-(P_n^r) \geq n-r$, except that, when $n-r$ is odd, we must have $\text{mr}^-(P_n^r) \geq n-r+1$. In particular, note that if $r = n-3$ or $r = n-4$, then $\text{mr}^-(P_n^r) \geq 4$.

For the cases $1 \leq r \leq n-3$, we proceed by induction on $n \geq 4$. For $n = 4$, we have $\text{mr}^-(P_4) = 4 = (4-1) + 1$, and $\text{mr}^-(P_4^2) = 2$. For $n = 5$, we have $\text{mr}^-(P_5) = 4 = 5-1$. Clearly (Figure 1), $\text{mr}^-(P_5^2) = \text{mr}^-(P_4) = 4 = 5-2+1$, and from Theorem 1.5, $\text{mr}^-(P_5^3) = 2$.

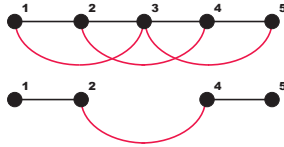


Figure 1: The graphs P_5^2 and $P_5^2 - \{3\}$.

Let k be a fixed integer, and suppose that for $4 \leq n \leq k-1$,

$$\text{mr}^-(P_n^r) = \begin{cases} n-r & \text{if } 1 \leq r \leq n-3 \text{ and } n-r \text{ is even,} \\ n-r+1 & \text{if } 1 \leq r \leq n-3 \text{ and } n-r \text{ is odd,} \\ 2 & \text{if } r \geq n-2. \end{cases}$$

Let $n = k$, and let r be an integer such that $1 \leq r \leq n-3$. Let H_1 be the subgraph of P_n^r , induced by the set of $r+2$ vertices $\{1, 2, \dots, r+2\}$, and H_2 the subgraph of

P_n^r , induced by the set of $n-2$ vertices $\{3, 4, \dots, n\}$, so $H_1 \cong P_{r+2}^r$, and $H_2 \cong P_{n-2}^r$, and keeping the original labels on the vertices

$$P_n^r \cong (H_1 \cup \{r+3, \dots, n-1, n\}) \bigcup (\{1, 2\} \cup H_2).$$

By 3, in Observation 1.3,

$$\begin{aligned} \text{mr}^-(P_n^r) &\leq \text{mr}^-(H_1 \cup \{r+3, \dots, n-1, n\}) + \text{mr}^-(\{1, 2\} \cup H_2) = \\ &\text{mr}^-(H_1) + \text{mr}^-(H_2) = \text{mr}^-(P_{r+2}^r) + \text{mr}^-(P_{n-2}^r). \end{aligned}$$

By previous discussion, $\text{mr}^-(P_{r+2}^r) = 2$, and by the induction hypothesis,

$$\text{mr}^-(P_{n-2}^r) = \begin{cases} n-r-2 & \text{if } 1 \leq r \leq n-5 \text{ and } n-r-2 \text{ is even,} \\ n-r-1 & \text{if } 1 \leq r \leq n-5 \text{ and } n-r-2 \text{ is odd,} \\ 2 & \text{if } r \geq n-4. \end{cases}$$

Since n and $n-2$ are both even or both odd, it follows that

$$\text{mr}^-(P_n^r) \leq 2 + \begin{cases} n-r-2 & \text{if } 1 \leq r \leq n-5 \text{ and } n-r \text{ is even,} \\ n-r-1 & \text{if } 1 \leq r \leq n-5 \text{ and } n-r \text{ is odd,} \\ 2 & \text{if } r \geq n-4, \end{cases}$$

and consequently, that

$$\text{mr}^-(P_n^r) \leq \begin{cases} n-r & \text{if } 1 \leq r \leq n-5 \text{ and } n-r \text{ is even,} \\ n-r+1 & \text{if } 1 \leq r \leq n-5 \text{ and } n-r \text{ is odd,} \\ 4 & \text{if } r \geq n-4. \end{cases}$$

Since $\text{mr}^-(P_n^r) \geq 4$ for $r = n-3$ or $r = n-4$, this is equivalent to

$$\text{mr}^-(P_n^r) \leq \begin{cases} n-r & \text{if } 1 \leq r \leq n-3 \text{ and } n-r \text{ is even,} \\ n-r+1 & \text{if } 1 \leq r \leq n-3 \text{ and } n-r \text{ is odd} \\ 4 & \text{if } r \geq n-2 \end{cases}$$

The proof of the theorem is now complete. \square

Corollary 2.3. *If n and r are positive integers, with $n \geq 3$, then $\text{mr}(P_n^r) \leq \text{mr}^-(P_n^r)$, with equality if and only if $1 \leq r \leq n-3$ and $n-r$ is even.*

2.2 Minimum Skew Rank of Strict Powers of Paths

Note that $P_2^{(r)} \cong P_2 \cong K_2$, if r is odd and $P_2^{(r)} \cong K_1 \cup K_1$, if r is even, thus when $r \geq 1$, $\text{mr}^-(P_2^{(r)}) = 2$ for r odd, $\text{mr}^-(P_2^{(r)}) = 0$ for r even. Also, $P_3^{(r)} \cong P_3$ if r is odd and $P_3^{(r)} \cong K_2 \cup K_1$ if r is even, thus for $r \geq 1$, $\text{mr}^-(P_3^{(r)}) = 2$.

Lemma 2.4. *Let n and r be positive integers, with $n \geq 3$ and $r \geq n - 2$.*

1. *If r is odd, then $\text{mr}^-(P_n^{(r)}) = 2$.*
2. *If r is even, then $\text{mr}^-(P_n^{(r)}) = 4$.*

Proof. Recall ([9]) that when $r \geq n - 2$, and r is odd, $P_n^{(r)} \cong K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, and when r is even, $P_n^{(r)} \cong K_{\lfloor n/2 \rfloor} \cup K_{\lceil n/2 \rceil}$. The result follows from Theorem 1.5. \square

Theorem 2.5. *If n and r are positive integers, with $n \geq 3$ and r odd, then*

$$\text{mr}^-(P_n^{(r)}) = \begin{cases} n - r & \text{if } 1 \leq r \leq n - 3 \text{ and } n \text{ is odd,} \\ n - r + 1 & \text{if } 1 \leq r \leq n - 3, \text{ and } n \text{ is even,} \\ 2 & \text{if } r \geq n - 2. \end{cases}$$

Proof. The case $r \geq n - 2$ follows from Lemma 2.4. Let $1 \leq r \leq n - 3$, and observe that each matrix in $\mathcal{S}^-(P_n^{(r)})$ is the sum of $r + 1$ band matrices (with zero diagonal) of bandwidths $1, 3, \dots, r + 1$ if r is even and bandwidths $2, 4, \dots, r + 1$ if r is odd, thus the upper left $(n - r) \times (n - r)$ submatrix is lower-triangular with nonzero diagonal ([14]), thus the upper left $(n - r) \times (n - r)$ submatrix is lower-triangular with nonzero diagonal, and we have $\text{mr}^-(P_n^{(r)}) \geq n - r$, except that, when $n - r$ is odd, we must have $\text{mr}^-(P_n^{(r)}) \geq n - r + 1$. In particular, note that if $r = n - 3$ or $r = n - 4$, then $\text{mr}^-(P_n^{(r)}) \geq 4$.

$$\text{Thus, } \text{mr}^-(P_n^{(r)}) \geq \begin{cases} n - r & \text{if } 1 \leq r \leq n - 3 \text{ and } n \text{ is odd,} \\ n - r + 1 & \text{if } 1 \leq r \leq n - 3 \text{ and } n \text{ is even} \\ 2 & \text{if } r \geq n - 2. \end{cases}$$

The rest of the proof is by induction on n , note that the cases $n = 3$ and $r \geq n - 2$ follow from Lemma 2.4. Let $1 \leq r \leq n - 3$ and assume that for $3 \leq n < k - 1$ we

$$\text{have } \text{mr}^-(P_n^{(r)}) = \begin{cases} n - r & \text{if } 1 \leq r \leq n - 3, \quad n \text{ is odd,} \\ n - r + 1 & \text{if } 1 \leq r \leq n - 3, \quad n \text{ is even} \\ 2 & \text{if } r \geq n - 2. \end{cases}$$

Let $n = k$, and let H_1 be the subgraph of $P_n^{(r)}$ induced by the set of vertices $\{1, 2, \dots, n - 2\}$, and H_2 the subgraph of $P_n^{(r)}$ induced by the set of vertices $\{n - r - 1, n - r, \dots, n\}$, so $H_1 \cong P_{n-2}^{(r)}$, and $H_2 \cong P_{r+2}^{(r)}$, and keeping the original labels on the vertices

$$P_n^{(r)} \cong (H_1 \cup \{n - 1, n\}) \bigcup (\{1, 2, \dots, n - r - 2\} \cup H_2).$$

By 3, in Observation 1.3,

$$\begin{aligned} \text{mr}^-(P_n^{(r)}) &\leq \text{mr}^-(H_1 \cup \{n-1, n\}) + \text{mr}^-(\{1, 2, \dots, n-r-2\} \cup H_2) = \\ &\text{mr}^-(H_1) + \text{mr}^-(H_2) = \text{mr}^-(P_{n-2}^{(r)}) + \text{mr}^-(P_{r+2}^{(r)}). \end{aligned}$$

So by Lemma 2.4 and the induction hypothesis,

$$\text{mr}^-(P_{n-2}^{(r)}) = \begin{cases} (n-2)-r & \text{if } 1 \leq r \leq (n-2)-3 \text{ and } n \text{ is odd,} \\ (n-2)-r+1 & \text{if } 1 \leq r \leq (n-2)-3 \text{ and } n \text{ is even,} \\ 2 & \text{if } r \geq (n-2)-2 \end{cases}$$

and $\text{mr}^-(P_{r+2}^{(r)}) = 2$. It follows that

$$\text{mr}^-(P_n^{(r)}) \leq \begin{cases} (n-2-r)+2 & \text{if } 1 \leq r \leq n-3 \text{ and } n \text{ is odd,} \\ (n-2-r+1)+2 & \text{if } 1 \leq r \leq n-3 \text{ and } n \text{ is even,} \\ 4 & \text{if } r \geq n-2 \end{cases}$$

$$\text{and hence, that } \text{mr}^-(P_n^{(r)}) = \begin{cases} n-r & \text{if } 1 \leq r \leq n-3 \text{ and } n \text{ is odd,} \\ n-r+1 & \text{if } 1 \leq r \leq n-3 \text{ and } n \text{ is even,} \\ 2 & \text{if } r \geq n-2. \end{cases} \quad \square$$

Lemma 2.6. *If n and m are positive integers with $n \geq 3$, then $P_n^{(2m)} \cong P_{\lfloor \frac{n}{2} \rfloor}^m \cup P_{\lceil \frac{n}{2} \rceil}^m$, the union being a disjoint union.*

Proof. As before, we denote the set of vertices of P_n by $\{1, 2, \dots, n-1, n\}$, the set of vertices of $P_{\lfloor \frac{n}{2} \rfloor}$ by $\{v_1, v_2, \dots, v_{\lfloor \frac{n}{2} \rfloor}\}$, and the set of vertices of $P_{\lceil \frac{n}{2} \rceil}$ by $\{u_1, u_2, \dots, u_{\lceil \frac{n}{2} \rceil}\}$.

Define $\phi : \{1, 2, \dots, n-1, n\} \longrightarrow \{v_1, v_2, \dots, v_{\lfloor \frac{n}{2} \rfloor}\} \cup \{u_1, u_2, \dots, u_{\lceil \frac{n}{2} \rceil}\}$ by

$$\phi(w) = \begin{cases} v_t & \text{if } w = 2t, t \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\} \\ u_s & \text{if } w = 2s-1, s \in \{1, 2, \dots, \lceil \frac{n}{2} \rceil\}. \end{cases}$$

Clearly ϕ is an bijection.

We know, from [9, Theorem 3.6] (see also Theorem 8.1.3 in [14]), that $uv \in E_{P_n^{(2m)}}$ if and only if $|u-v| \in \{2m, 2m-2, 2m-4, \dots, 2\}$, thus both u and v must be even, or both must be odd. Define $\psi : E_{P_n^{(2m)}} \longrightarrow E_{P_{\lfloor \frac{n}{2} \rfloor}^m} \cup E_{P_{\lceil \frac{n}{2} \rceil}^m}$ by

$$\psi(w_1 w_2) = \begin{cases} v_{t_1} v_{t_2} & \text{if } w_1 = 2t_1, w_2 = 2t_2, t_1, t_2 \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\} \\ u_{s_1} u_{s_2} & \text{if } w_1 = 2s_1-1, w_2 = 2s_2-1, s_1, s_2 \in \{1, 2, \dots, \lceil \frac{n}{2} \rceil\}. \end{cases}$$

Assume that $\psi(w_1 w_2) = v_{t_1} v_{t_2}$, for some $t_1, t_2 \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$, then $|v_{t_1} - v_{t_2}| = |t_1 - t_2| = \frac{1}{2}|w_1 - w_2| \in \{m, m-1, m-2, \dots, 1\}$, thus $0 < |v_{t_1} - v_{t_2}| \leq m$ and $v_{t_1} v_{t_2} \in E_{P_{\lfloor \frac{n}{2} \rfloor}^m}$. Similarly, $\psi(w_1 w_2) = u_{s_1} u_{s_2} \in E_{P_{\lceil \frac{n}{2} \rceil}^m}$. The details for showing ψ is an bijection are straightforward. \square

Theorem 2.7. *If n and r are positive integers, with $n \geq 4$, and $r = 2s$, then*

$$\text{mr}^-(P_n^{(r)}) = \begin{cases} n - r + 1 & \text{if } 1 \leq r \leq n - 3 \text{ and } n \text{ is odd,} \\ n - r & \text{if } 1 \leq r \leq n - 3, n = 2t, t - s \text{ even} \\ n - r + 2 & \text{if } 1 \leq r \leq n - 3, n = 2t, t - s \text{ odd} \\ 4 & \text{if } r \geq n - 2. \end{cases}$$

Proof. The case $r \geq n - 2$ follows from Lemma 2.4. Suppose n is odd, and $1 \leq r \leq n - 3$. By Lemma 2.6, $P_n^{(r)} = P_n^{(2s)} \cong P_{\lceil \frac{n}{2} \rceil}^s \cup P_{\lfloor \frac{n}{2} \rfloor}^s$, and one of $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ is even and the other is odd. In either case, from Theorem 2.2, and 2 in Observation 1.3, it follows that $\text{mr}^-(P_n^{(r)}) = \text{mr}^-(P_{\lceil \frac{n}{2} \rceil}^s \cup P_{\lfloor \frac{n}{2} \rfloor}^s) = \text{mr}^-(P_{\lceil \frac{n}{2} \rceil}^s) + \text{mr}^-(P_{\lfloor \frac{n}{2} \rfloor}^s) = \lceil \frac{n}{2} \rceil - s + \lfloor \frac{n}{2} \rfloor - s + 1 = n - 2s + 1 = n - r + 1$.

Now let n be even, $n = 2t$, and $1 \leq r \leq n - 3$, so that by Lemma 2.6, $P_n^{(r)} = P_n^{(2s)} \cong P_t^s \cup P_t^s$, and thus $\text{mr}^-(P_n^{(r)}) = 2 \text{mr}^-(P_t^s)$. From Theorem 2.2, $\text{mr}^-(P_t^s) = t - s$, if $t - s$ is even, and $t - s + 1$, if $t - s$ is odd. Therefore, $\text{mr}^-(P_n^{(r)}) = n - r$, if $t - s$ is even, and $n - r + 2$, if $t - s$ is odd. \square

Corollary 2.8. *If n and r are positive integers with $n \geq 3$, then $\text{mr}(P_n^{(r)}) \leq \text{mr}^-(P_n^{(r)})$, with equality if and only if n and r are both odd, or n and r are both even, $n = 2t$, $r = 2s$ and $t - s$ is even.*

References

- [1] American Institute of Mathematics workshop, Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns, held October 23–27, 2006 in Palo Alto, CA.
- [2] M. Allison, E. Bodine, L. M. DeAlba, J. Debnath, L. DeLoss, C. Garnett, J. Grout, L. Hogben, B. Im, H. Kim, R. Nair, O. Pryporova, K. Savage, B. Shader, A. Wangsness Wehe. Minimum rank of skew-symmetric matrices described by a graph. *Linear Algebra and its Applications* 432 (2010) 2457–2472.
- [3] F. Barioli, S. Fallat, and L. Hogben. Computation of minimal rank and path cover number for graphs. *Linear Algebra and its Applications* 392:289–303, 2004.
- [4] F. Barioli, S. M. Fallat, D. Hershkowitz, H. T. Hall, L. Hogben, H. van der Holst, B. Shader. On the minimum rank of not necessarily symmetric matrices: a preliminary study. *The Electronic Journal of Linear Algebra* 18 (2009) 126145.

- [5] F. Barioli, S. M. Fallat, R. L. Smith. On acyclic and unicyclic graphs whose minimum rank equals the diameter. *Linear Algebra and its Applications* 429 (2008) 15681578.
- [6] W. Barrett, H. van der Holst, R. Loewy. Graphs whose minimal rank is two. *The Electronic Journal of Linear Algebra* 11 (2004) 258280.
- [7] W. Barrett, H. van der Holst, R. Loewy. Graphs whose minimal rank is two: the finite fields case. *The Electronic Journal of Linear Algebra* 14 (2005) 3242.
- [8] W. Barrett, J. Grout, R. Loewy. The minimum rank problem over the finite field of order 2: minimum rank 3. *Linear Algebra and its Applications* 430 (2009) 890923.
- [9] L. M. DeAlba, J. Grout, I. J. Kim, S. Kirkland, J. J. McDonald, A. Yielding. Minimum rank of powers of graphs. Preprint.
- [10] L. M. DeAlba, J. Grout, L. Hogben, R. Mikkelsen, and K. Rasmussen. Universally optimal matrices and field independence of the minimum rank of a graph. *The Electronic Journal of Linear Algebra* 18:403–419, 2009.
- [11] L. M. DeAlba. Acyclic and unicyclic graphs whose minimum skew rank is equal to the minimum skew rank of a diametrical path. arXiv:1107.2170v1.
- [12] Reinhard Diestel. *Graph Theory*, Third Edition, Springer, 2000.
- [13] S. Fallat and L. Hogben. The minimum rank of symmetric matrices described by a graph: A survey. *Linear Algebra and Its Applications* 426/2–3 (2007) 558–582.
- [14] Miroslav Fiedler. *Special Matrices and Their Applications in Numerical Mathematics*, Second Edition, Dover, 2008.
- [15] L. Hogben. Minimum rank problems. *Linear Algebra and its Applications* doi:10.1016/j.laa.2009.05.003
- [16] Institute of Mathematics and its Applications PI Summer Program: Linear Algebra and Applications. Iowa State University, Ames, Iowa. June 30–July 25, 2008.
- [17] Douglas West. *Introduction to Graph Theory*, Second Edition, Prentice Hall, 2001.